The Temperly-Lieb algebra:

generated by E_i with $i=1,\dots,n-1$ satisfying

$$E_i E_j = E_j E_i \,, \quad \text{for } |i-j| \geq 2,$$

$$E_i E_{i\pm 1} E_i = E_i,$$

$$E_i^2 = d E_i$$

Next, want to embed the braid group B_n into $TL_n(d)$

$\to$ define $\rho_A(\sigma_i) = A E_i + A^{-1} \mathbb{1}$

and $\rho_A(\sigma_i^{-1}) = A^{-1} E_i + A \mathbb{1}$

Then check:

1) $$\rho_A(\sigma_i)\,\rho_A(\sigma_i^{-1}) = (A E_i + A^{-1}\mathbb{1})(A^{-1}E_i + A\mathbb{1})$$
$$= E_i^2 + A^2 E_i + A^{-2} E_i + \mathbb{1}$$
$$= dE_i - dE_i + \mathbb{1} = \mathbb{1} \quad \checkmark$$

2) For $|i-j| \geq 2$ compute
$$\rho_A(\sigma_i)\rho_A(\sigma_j) = (A E_i + A^{-1}\mathbb{1})(A E_j + A^{-1}\mathbb{1})$$
$$= (A E_j + A^{-1}\mathbb{1})(A E_i + A^{-1}\mathbb{1})$$
$$= \rho_A(\sigma_j)\,\rho_A(\sigma_i) \quad \checkmark$$

3) $$\rho_A(\sigma_i)\rho_A(\sigma_{i+1})\rho_A(\sigma_i)$$
$$= A^3 E_i E_{i+1} E_i + A E_{i+1} E_i + A E_i^2 + A^{-1} E_i$$

$$+ A E_i E_{i+1} + A^{-1} E_{i+1} + A^{-1} E_i + A^3$$

$$= (A^3 + Ad + A^{-1}) E_i + 2 A E_i E_{i+1} + A^{-1}(E_i + E_{i+1}) + A^3$$

$$= \rho_A(\sigma_i) \rho_A(\sigma_{i+1}) \rho_A(\sigma_i)$$

where we used $A^3 + Ad + A^{-1} = 0$

$\rightarrow$ $\rho(\sigma_i)$ is a representation of the braid group generators.

If we take $|A| = 1$ and E_i Hermitian for all i, we get

$$\rho(\sigma_i)\rho(\sigma_i)^\dagger = (A E_i + A^{-1} \mathbb{1})(A^* E_i^\dagger + (A^{-1})^* \mathbb{1})$$

$$= E_i^2 + (A^2 + A^{-2}) E_i$$

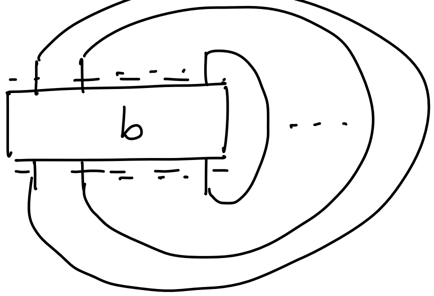
$$= \mathbb{1}$$

$\rightarrow$ $\rho(\sigma_i)$ is unitary!

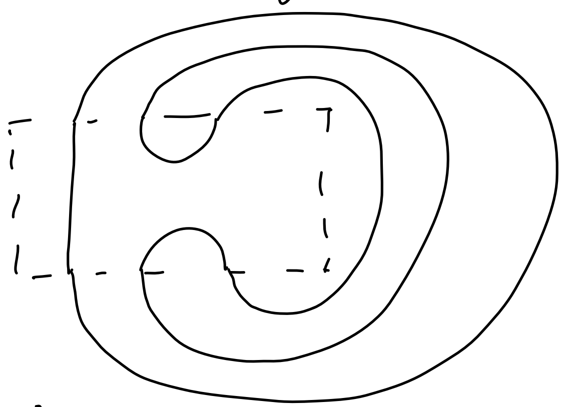
Markov trace and Jones polynomials

The Markov trace is defined as taking the trace over the representation $\rho_A(b)$ of a braid word b $\rightarrow$ number (polynomial in A)

$\rightarrow$ pictorially, this number corresponds to a link $L = (b)^{\text{Markov}}$:



We denote the Markov trace of a product of elements of $TL_n(d)$ by K

$\longrightarrow \quad tr(K) =$ 

Kauffman diagram

$$:= d^{a-n}$$

where n = # points at hor. side of K

a = # loops

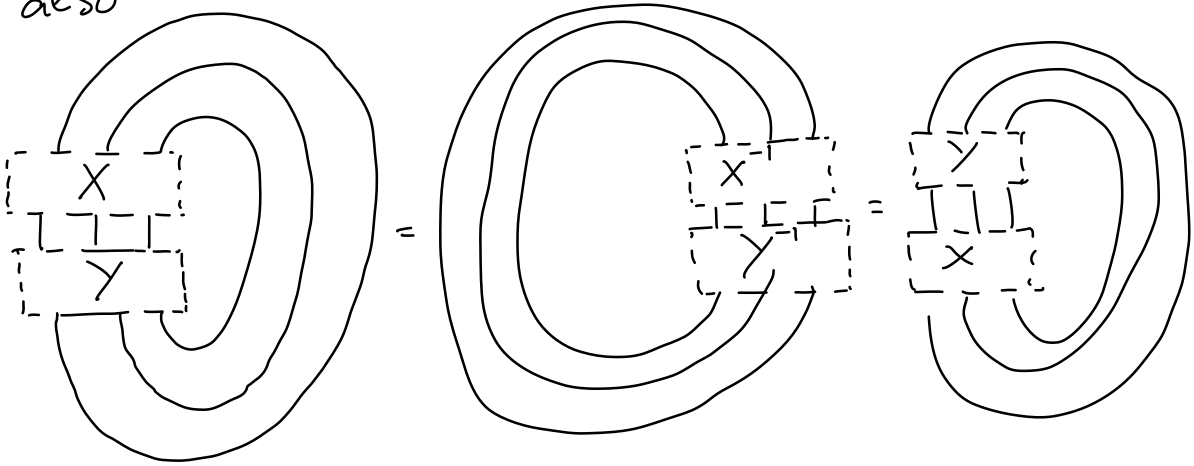
Markov trace has following properties:

a) $tr(\mathbb{1}) = 1$,

b) $tr(XY) = tr(YX)$, for any $X, Y \in TL_n(d)$

c) $tr(XE_{n-1}) = \frac{1}{d} tr(X)$, for any $X \in TL_{n-1}(d)$

Indeed, the trace of the Kauffmann diagram of $\mathbb{1}$ gives $a = n$ loops $\rightarrow$ a) also



$\longrightarrow$ b)

At last, removing a single E_{n-1} element produces a loop $\longrightarrow$ c)

Have the following identity:

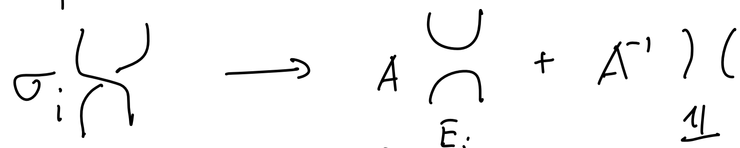
$$d^{n-1}\, tr(\rho_A(b)) = \langle (b)^{Markov} \rangle (A)$$

$$= \langle L \rangle := \sum_{\{S\}} d^{|S|-1} A^{s^+ - s^-}$$

2^N configurations where $N = \#$ crossings

Proof:

This follows from

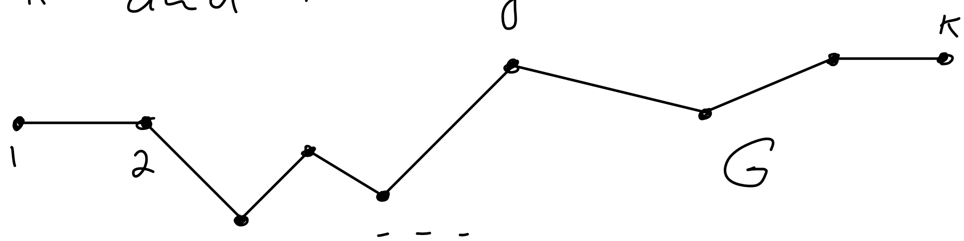


$tr(\rho_A(b))$ produces d^{a-n} which together with d^{n-1} gives $d^{a-1} = d^{|S|-1}$ as $|S| = a$ $\square$

$\rightarrow$ write Jones pol as $V_L(A) = (-A)^{3w(L)} d^{n-1}\, tr(\rho_A(b))$

Path-model representation of TL algebra

Suppose G is a one-dimensional graph with k vertices labeled by $1, \dots, k$ and $k-1$ edges.



→ consider an n step walk on G starting from left endpoint

→ denote path by $p = (\nu_1, \dots, \nu_n)$ where $\nu_1 = 1$ and $\underset{\text{move left}}{\nu_j = 0}$ or $\underset{\text{move right}}{\nu_j = 1}$

→ define Hilbert space $\mathcal{H}_{n,k}$ spanned by all possible paths as a basis $\{|p\rangle\}$.

For each generator $E_i \in TL_n(d)$ define representation $\Phi_i = \rho(E_i)$ as follows:

$$\Phi_i \,|\cdots \nu_{i-1}\, 0 0\, \nu_{i+1} \cdots\rangle = 0,$$

$$\Phi_i \,|\cdots \nu_{i-1}\, 0 1\, \nu_{i+1} \cdots\rangle = \frac{\lambda_{z_i-1}}{\lambda_{z_i}} |\cdots \nu_{i-1} 0 1 \nu_{i+1}\cdots\rangle + \frac{\sqrt{\lambda_{z_i+1}\lambda_{z_i-1}}}{\lambda_{z_i}} |\cdots \nu_{i-1} 1 0 \nu_{i+1}\cdots\rangle$$

$$\Phi_i \,|\cdots \nu_{i-1}\, 1 0\, \nu_{i+1} \cdots\rangle = \frac{\lambda_{z_i+1}}{\lambda_{z_i}} |\cdots \nu_{i-1} 1 0 \nu_{i+1}\cdots\rangle + \frac{\sqrt{\lambda_{z_i+1}\lambda_{z_i-1}}}{\lambda_{z_i}} |\cdots \nu_{i-1} 0 1 \nu_{i+1}\cdots\rangle$$

$$\Phi_i \,|\cdots \nu_{i-1}\, 1 1\, \nu_{i+1} \cdots\rangle = 0,$$

where $z_i \in \{1, \cdots, k\}$ is the vertex label at the ith step and $\lambda_j = \sin(j\theta)$ with $\theta = \pi/k$.

$\longrightarrow$ induced representation
$\tilde{\rho}(\sigma_i) = A\,\Phi_i + A^{-1}\mathbb{1}$ of braid group B_n

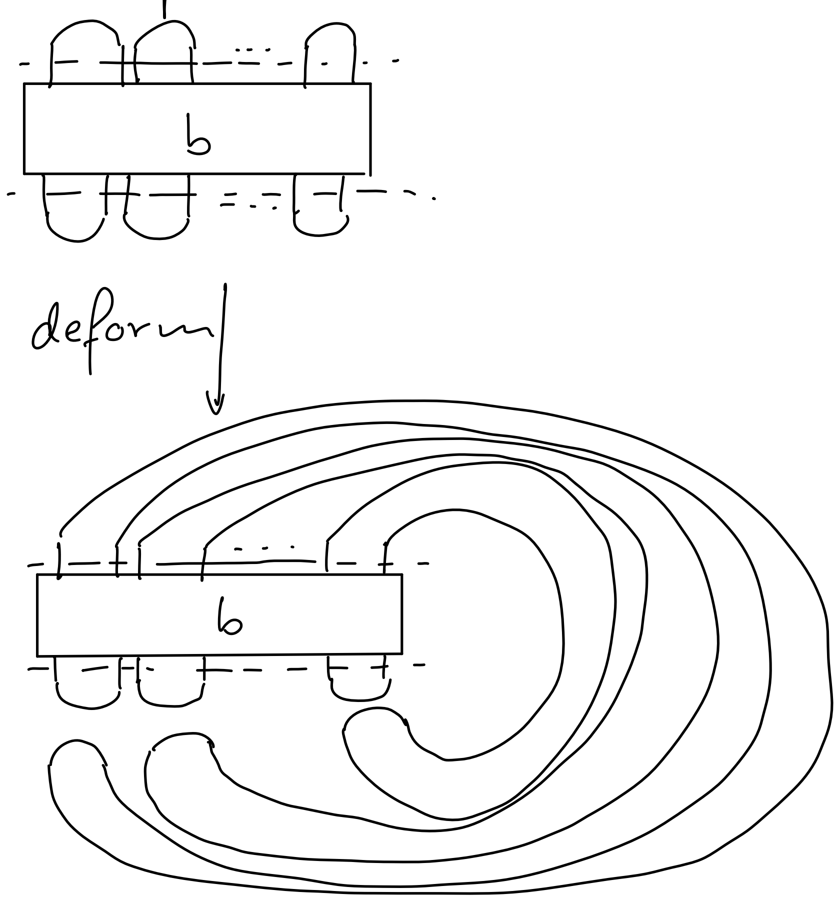
Setting $A = ie^{-i\theta/2}$ and $d = 2\cos\theta$ gives hermitian E_i for all i

$\rightarrow$ $\tilde{\rho}(\sigma_i)$ is unitary for this choice acting on neighboring two-qubit

→ can be implemented by universal quantum computer

By the Hadamard test, the Jones polynomial can be evaluated as a trace with an additive error $\Delta 2^n \varepsilon$ taking $\mathrm{poly}(n, m, k, 1/\varepsilon)$ overhead, where m is # crossings.

Taking the plat closure,



→ the support of $\rho(b)$ is only on $|10101\cdots\rangle$

→ replace matrix trace by

$$V_{b^{pef}}(A^{-4}) = \Delta_{pef} \langle 10101\cdots|\rho(b)|10101\cdots\rangle$$

where b^{pef} is a link generated from braiding b with plat closure

and $\Delta_{pef} = (-A)^{3w(b^{pef})} d^{n/2-1}$

$\xrightarrow{\text{Hadamard test}}$ estimate $V_{b^{pef}}(A^{-4})$ with additive error $\Delta_{pef}\,\varepsilon$ taking $\mathrm{poly}(n, m, k, \frac{1}{\varepsilon})$ time